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Multifractality and the shattering transition in fragmentation processes

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We consider two simple geometric models that can describe the kinetics of fragmentation of two-dimensional particles and stochastic fractals. We find a hierarchy of independent exponents suggesting the existence of multiple-phase boundary for the shattering transition when two orthogonal cracks are placed randomly on a fragment (model A). At the same time we find a unique exponent suggesting a single phase boundary when four equal-sized fragments are produced at each fragmentation event (model B). We invoke the multifractal formalism to further support the existence of multiple phase boundaries. In model A, for each choice of homogeneity index, the resultant fragments' distribution exhibits multifractality on a unique support when describing fragmentation processes and on one of infinitely many possible supports when describing stochastic fractals. Model B obeys simple scaling and produces self-similar fractals when fragments are removed from the system at each time step. [S1063-651X(96)02508-1]

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I. INTRODUCTION

Fragmentation is a phenomenon that occurs in numerous physical, chemical, and geological processes. Recently, it has been of considerable interest [1–4]. In general, it is an irreversible kinetic process in which a collection of fragments is sequentially broken. Although conceptually it is quite simple to understand, its kinetic and many other physical quantities are not fully understood, especially in higher dimensions. There have been a number of different approaches to understanding the fragmentation phenomena of one-dimensional particles analytically. These include using the maximum entropy method [5], using statistical and combinatorial arguments [6,7], and using a kinetic equation. It is the kinetic equation approach, developed by Filippov [8] after its original proposal by Kolmogorov [9], that has provided much of our theoretical understanding. Interestingly, it has also been used successfully to describe other phenomena; for example, random sequential adsorption [10,11] and stochastic fractals [12,13].

In one dimension the fragmentation process is well studied with a large class of exact and explicit solutions for the particle-size distribution function using different fragmentation rules [14,15], in addition to scaling solutions [16,17]. The scaling solutions are of considerable importance since most experimental systems reach this behavior in the long time. These are essentially the solutions in the long-time ($t\to\infty$) and small-size ($x\to 0$) limit, where the probability-distribution function evolves into a simpler form since it reduces the two variable problems to a single variable (in one dimension). Moreover, this form is universal in the sense that it does not depend on the initial condition. A number of interesting features have been found in one-dimensional problems, such as a shattering transition that is accompanied by the violation of scaling and absence of self-averaging [18]. However, recently much effort has been devoted to higher-dimensional problems where particles are characterized by both size and shape, unlike in one dimension where size or mass is the only dynamical quantity of interest. This is motivated by the desire to move towards an understanding of the physical role played by shape in the fragmenting systems, since in reality particles are identified by their size and shape. Studying the fragmentation phenomena in higher dimensions has revealed interesting and nontrivial features [2,4] with unexpectedly rich patterns of fragments.

In higher dimensions the scaling regime does not reduce to a single variable, but more than one intriguing variable, which causes the system to show multiscaling [2]. This particular feature is the signature of large fluctuations and causes the absence of self-averaging. One can immediately anticipate the occurrence of multifractality that became essential in recent years, to get a deeper insight into the structure of such a wildly varying system. Multifractal phenomena have become very active in the research area and are found to describe many physical systems in different contexts. These include voltage and current distribution in random resistor networks [26], growth by diffusion-limited aggregation (DLA) [27], collision cascade [28], and percolation and fracture [29]. The basic idea of multifractality is that, given a fragments' distribution on a measure support characterized by a set of points (D_f), a richer structure can be invoked. Namely, the whole set (D_f) can be partioned into a hierarchy of subsets with their own fractal dimensions $f(\alpha)$. The spectrum of these dimensions gives the full characterization of the object. Different clusters corresponding to their fractal subsets are scaled with their own exponents.

The general form of the fragmentation equation when a given d-dimensional particle fragments into 2^d pieces per fragmentation event is given by

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$$\frac{\partial f(\{x_i\};t)}{\partial t} = -f(\{x_i\};t)\int_0^{\{x_i\}}\prod_{i=1}^{d} dx_i' F(x_1-x_1',x_1';\ldots;x_d-x_d',x_d') + 2^d\int_{\{x_i\}}^{\infty}\prod_{i=1}^{d} dx_i' F(x_1'-x_1,x_1;\ldots;x_d'-x_d,x_d) f(\{x_i'\};t), \quad (1)$$

where $f(\{x_i\};t)$ is the probability-distribution function of a d-dimensional hypercuboid-shaped object of sides $\{x_i\}$ at time t and $F(x_1-x_1',x_1';\ldots;x_d-x_d',x_d')$ is the rate of fragmentation of a particle characterized by x_1, x_2,...,x_d into 2^d fragmented smaller particles characterized by their sides of (x_i-x_i') and x_i'. The d number of integrals on $\{x_i'\}$ variables is equivalent to placing d number of orthogonal cracks such that they are equal and parallel to their sides to produce 2^d fragments at each fragmentation event. Solving this general equation for some fragmentation rule is very difficult. However, for some simple choice of fragmentation rule, the explicit solution to this general problem is found in [4]. Obviously in reality the cracks should appear at random on the object to be fragmented to produce a large number of fragments with a varying number of sides. The present kinetic approach is a simplified version of this real picture; however, it has been found that this simple model can provide the basic physics of the fragmenting systems.

In this paper we restrict ourselves to two dimensions, which is the minimum dimension to exhibit the role played by shape. The rest of this paper is organized as follows. We attempt to investigate the shattering transition in order to explain the rich pattern formed in the long time. Our treatment is based on considering the asymptotic regime to obtain information about scaling. Through our approach a new feature appears: instead of a unique-phase boundary we find infinitely many phase boundaries because of the multiple conservation laws. This is in contrast to [3], where one variable was associated with energy and the other with mass and a single phase boundary was found. In order to clarify the origin of this behavior we invoked the idea of multifractality, which appears to require not one but infinitely many scaling exponents. We seek to develop a one-to-one correspondence between the existence of multiple phase boundaries and multiple scaling exponents. We also find that some fragmentation rules do not yield a single measure support, instead they have an infinite number for each choice of homogeneity index. We consider a second model in two dimensions and we seek to explain the difference between these models.

II. SHATTERING TRANSITION

The fragmentation equation in two dimensions is

$$\frac{\partial f(x,y;t)}{\partial t} = -f(x,y;t)\int_0^x dx_1\int_0^y dy_1 F(x_1-x,x_1;y_1-y,y_1) + s\int_x^{\infty} dx_1\int_y^{\infty} dy_1 f(x_1,y_1;t) \times F(x,x_1-x;y,y_1-y), \quad (2)$$

where $s=1,2,3,4$ and $F(x_1,x-x_1;y_1,y-y_1)$ describes the rate at which objects having sides x and y break to produce fragments of sides (x_1,y_1), $(x-x_1,y_1)$ $(x_1,y-y_1)$, and $(x-x_1,y-y_1)$. This two-dimensional problem can be viewed as orthogonal, cross-shaped cracks are placed on the fragments that remain fixed at their spatial position, but the cracks are placed on objects to be fragmented homogeneously. However, the way the cracks are to be placed is determined by the choice of the kernel. Alternatively, we can assume that there is a perfect mixing of fragments that makes this kinetic approach so appealing to describe many physical systems, including the ball-milling and batch grinding of the comminution process. However, we can choose, for example, $s=1,2,3$, which simply implies the removal of $4-s$ of the fragments at each event to form a stochastic fractal [12,13] at long times. We define the moment of the distribution function as

$$M_{m,n}(t) = \int_0^{\infty} x^{m-1}dx\int_0^{\infty} y^{n-1}dy\ f(x,y;t), \quad (3)$$

which is the double Mellin transform of the probability distribution function $f(x,y;t)$, where $m,n>0$ is required for convergence, and where $M_{1,1}(t)$ is the number of fragments in the system at any time t and $M_{2,2}(t)$ is the total area of the fragments. Since moments keep the signature of the distribution function, it is easy to deal with in some cases, to get the main features of the fragmenting systems.

A. Model *A*

We choose to study a homogeneous rate kernel [3,21],

$$F(x_1,x_2;y_1,y_2) = (x_1+x_2)^{\beta_1}(y_1+y_2)^{\beta_2}, \quad (4)$$

where β_1 and β_2 are known as the homogeneity indices. This choice of fragmentation implies that the distribution of particles upon breakage depends only on the ratio of the size of the broken and breaking particles. The breakup time τ is defined as

$$\frac{1}{\tau(x,y)} = \int_0^x dx_1\int_0^y dy_1 F(x_1,x-x_1;y_1,y-y_1) = x^{\beta_1+1}y^{\beta_2+1}. \quad (5)$$

This immediately confirms that for a certain choice of β_1 and β_2 the fragmentation process can be fast enough such that subsequent generation of fragments has a shorter lifetime than the previous generation. In this case shattering is anticipated in which mass is lost to a dust of zero-size particles and is identified by the singularity of the kinetic exponent [11,15].

The model in question describes a system in which particles are selected for fragmentation with a rate determined by their area and shape. The relative importance of area and shape is determined by their homogeneity indices β_1 and β_2. That is, if $\beta_1>\beta_2$, for example, different particles with equal area no longer compete on an equal footing to be fragmented, but rectangles with longer sides along the x axis are more likely to be picked for fragmentation than others. Nevertheless, once a fragment has been chosen for fragmentation, products of any area and shape are equally likely to occur. Substituting this choice of kernel into the rate equation yields

$$\frac{\partial f(x,y,;t)}{\partial t}=-x^{\beta_1+1}y^{\beta_2+1}f(x,y;t)+s\int_x^\infty dx_1x_1^{\beta_1}\int_y^\infty dy_1y_1^{\beta_2}f(x_1,y_1;t). \quad (6)$$

We can substitute the definition of the moment into the above equation to obtain a rate equation for the moment,

$$\frac{\partial M_{m,n}(t)}{\partial t}=\left(\frac{s}{mn}-1\right)M_{m+\beta_1+1,n+\beta_2+1}. \quad (7)$$

An interesting feature of the above equation is that there are infinitely many conserved (time-independent) moments satisfying $mn=s$ (multiple conservation laws). This simply reflects the fact that fragments with a given area can have an infinite number of different shapes. We choose $m=m^*$, where m^* is any number; then the rate equation for the moment determines the n value to be s/m^*, such that $M_{m^*,s/m^*}(t)$ is time independent. As in the one-dimensional case we propose a general scaling ansatz as $t\to\infty$ to be

$$f(x_1,x_2,;t)\sim t^w\phi(x_1t^{z_1},x_2t^{z_2}), \quad (8)$$

where the z_i's are known as the kinetic exponents. This ansatz is only true when both z_1 and z_2 are positive. Substituting into the definition of the moment and insisting that the moment $M_{m^*,s/m^*}(t)$ be a conserved quantity, we immediately obtain

$$w=m^*z_1+\frac{s}{m^*}z_2. \quad (9)$$

For a fixed positive z_1 and z_2 the curve w against m^* is an upward convex in shape. This implies that for each w value there are two different m^* values, say m^* and $m^\dagger$, except when the gradient is zero. Using these two relations for w for two different m^* values $(m^*,m^\dagger)$, we can express z_1 in terms of z_2,

$$z_1=\frac{s}{m^*m^\dagger}z_2. \quad (10)$$

The above equation implies that the two numbers m^* and $m^\dagger$ corresponding to the same w value will determine the ratio of the z_1 and z_2 values. Note that, if z_1 and z_2 are of opposite sign it is not possible to have two numbers give the same w value, as one can see from the asymptotic behavior of w against m^*. Although it is possible when both are negative, we do not need to concern ourselves about it since scaling is not valid in this case.

We now substitute the scaling ansatz into the rate equation to get

$$t^{z_1(\beta_1+1)+z_2(\beta_2+1)-1}=\frac{-\xi^{\beta_1+1}\xi_2^{\beta_2+1}\phi(\xi_1,\xi_2)+s\int_{\xi_1}^\infty\eta_1^{\beta_1}d\eta_1\int_{\xi_2}^\infty d\eta_2\eta_2^{\beta_2}\phi(\eta_1,\eta_2)}{w\phi(\xi_1,\xi_2)+z_i\xi_i\dfrac{d\phi(\xi_1,\xi_2)}{d\xi_i}}. \quad (11)$$

We demand that a scaling or time-invariant solution exists, and this immediately yields

$$z_1(\beta_1+1)+z_2(\beta_2+1)=1. \quad (12)$$

Combining this with Eq. (10) we find

$$z_1=\frac{s}{s(\beta_1+1)+m^*m^\dagger(\beta_2+1)} \quad (13)$$

and

$$z_2=\frac{m^*m^\dagger}{s(\beta_1+1)+m^*m^\dagger(\beta_2+1)}. \quad (14)$$

Thus any two nonzero and unequal numbers can give a set of kinetic exponents to reveal that there exists a hierarchy of exponents. The shattering is identified by the singularity in these exponents and hence we have an infinite-number phase boundary in the (β_1,β_2) plane, and all the phase lines meet at $(-1,-1)$.

B. Model *B*

We shall now consider a different model by choosing the following homogeneous kernels:

$$F(x_1,x_2;y_1,y_2)=(x_1+x_2)^{\beta_1}(y_1+y_2)^{\beta_2}\delta(2x_1-x)\times\delta(2y_1-y). \quad (15)$$

This model describes the case when two orthogonal cracks are placed to produce four fragments of equal size and shape. Thus, it is just one of the infinitely many possible ways of placing cracks in the former model. The breakup time $\tau(x,y)=x^{-\beta_1}y^{-\beta_2}$ again confirms that for certain choices of homogeneity indices the processes can be fast enough to show the shattering transition. Using this kernel the rate equation becomes

$$\frac{\partial f(x,y;t)}{\partial t}=-\frac{1}{2^2}x^{\beta_1}y^{\beta_2}f(x,y;t)+2^{\beta_1+\beta_2}sx^{\beta_1}x^{\beta_2}f(2x,2y;t), \quad (16)$$

and substituting the definition of the moments into the rate equation we obtain the rate equation for the moments as

$$\frac{\partial M_{m,n}(t)}{\partial t} = -\left(\frac{1}{2^2} - \frac{s}{2^{m+n}}\right) M_{m+\beta_1,n+\beta_2}(t). \quad (17)$$

Notice that this again gives infinitely many hidden conserved quantities. That is, assuming $m = m^*$, any positive number as before, the above equation implies that $M_{m^*,2+\ln s/\ln 2-m^*}(t)$ is the time-independent quantity. We now substitute the scaling ansatz into the rate equation to obtain

$$t^{z_1\beta_1+z_2\beta_1-1} = \frac{\xi_1^{\beta_1}\xi_2^{\beta_2}\{-\frac{1}{4}\phi(\xi_1,\xi_2)+2^{\beta_1+\beta_2+2}\phi(2\xi_1,2\xi_2)\}}{w\phi(\xi_1,\xi_2)+z_i\xi_i \dfrac{d\phi(\xi_1,\xi_2)}{d\xi_i}}. \quad (18)$$

Insisting that a scaling exists gives

$$z_1\beta_1 + z_2\beta_2 = 1. \quad (19)$$

Substituting the scaling ansatz into the definition of the moment and insisting again that $M^{m^*}_{2+\ln s/\ln 2-m^*}(t)$ be conserved, we obtain

$$w = m^* z_1 + \left(2 + \frac{\ln s}{\ln 2} - m^*\right) z_2. \quad (20)$$

Choosing any other nonzero and unequal number, say $m^\dagger$ instead of m^*, yields

$$z_1 = z_2 \equiv z. \quad (21)$$

Combining this with Eq. (19) we immediately get

$$z = \frac{1}{\beta_1+\beta_2}. \quad (22)$$

We now attempt to solve the rate equation directly to find the kinetic exponent. We multiply the rate equation by t on both sides to get

$$t\,\frac{\partial f(x_1,x_2;t)}{\partial t} = -\frac{1}{2^2}\,\xi_1^{\beta_1}\xi_2^{\beta_2} f(x_1,x_2;t) + 2^2(2\xi_1)^{\beta_1}(2\xi_2)^{\beta_2} f(2x_1,2x_2;t), \quad (23)$$

where $\xi_i = x_i t^{1/(\beta_1+\beta_2)}$. In the limit $t \to 0$ and $x \to \infty$, such that $\xi_i \to$ constant quantity, one can solve the equation to give

$$f(x_1,x_2;t) \sim T(t) t^{-[1+4/(\beta_1+\beta_2)]} \phi(\xi_1,\xi_2). \quad (24)$$

Substituting this into the definition of the moment and using the condition that $M_{m^*,2+\ln s/\ln 2-m^*}(t)$ is time independent gives

$$f(x_1,x_2;t) \sim t^{4/(\beta_1+\beta_2)} \phi(\xi_1,\xi_2). \quad (25)$$

The same kinetic exponent has been found in Ref. [4] in which it has been derived from the explicit solution and proved that the singularity in this exponent leads to the shattering transition. The appearance of the unique kinetic exponent confirms that there is a single phase boundary for the shattering transition in the plane (β_1,β_2).

III. ASYMPTOTIC SOLUTIONS

A. Model *A*

We now invoke the idea of multifractality that can characterize the rich pattern of the resultant fragments' distribution in the long-time limit. Using Charlesby's method, the moment equation (7) can be iterated to get all the higher derivatives of the moments. These can then be substituted into a Taylor-series expansion of $M_{m,n}(t)$ about $t=0$ to find the solution of Eq. (7) in terms of generalized hypergeometric function [19],

$$M_{m,n}(t) = {}_2F_2\left(a_+, a_-; \frac{m}{\beta_1+1}, \frac{n}{\beta_2+1}; -t\right), \quad (26)$$

where, defining $G = 4s(\beta_1+1)(\beta_2+1)$, $a_\pm$ are given by

$$a_\pm = \frac{m}{2(\beta_1+1)} + \frac{n}{2(\beta_2+1)} \pm \sqrt{\frac{[m(\beta_2+1)-n(\beta_1+1)]^2+G}{4(\beta_1+1)^2(\beta_2+1)^2}}. \quad (27)$$

We are only interested in the long-time behavior of the moment. The asymptotic expansion of the generalized hypergeometric function for large time t gives

$$M_{m,n}(t) \approx \frac{\Gamma\left(\dfrac{m}{\beta_1+1}\right)\Gamma\left(\dfrac{n}{\beta_2+1}\right)\Gamma(a_- - a_+)}{\Gamma(a_+)\Gamma\left[\left(\dfrac{m}{\beta_1+1}\right)-a_-\right]\Gamma\left[\left(\dfrac{n}{\beta_2+1}\right)-a_-\right]}\, t^{-a_-}, \quad (28)$$

provided $\beta_1,\beta_2 \neq -1$ for which the moment does not show power-law behavior.

B. Model *B*

For this splitting model we already know that the system reaches a scaling regime with the kinetic exponent given by Eq. (22). Knowledge of this information is sufficient to write the asymptotic solution for the moment, provided $\beta_1+\beta_2>0$ is as

$$M_{m,n}(t) \sim A(m,n) t^{-[m+n-(2+\ln s/\ln 2)]/(\beta_1+\beta_2)}, \quad (29)$$

where, $2+\ln s/\ln 2$ is determined by using hidden conserved quantities for which the moment becomes time independent. This solution has been derived in Ref. [21] explicitly using different methods.

IV. FRACTAL DIMENSIONS

A. Model *A*

When $1<s<4$, at each time step $4-s$ fragments are removed from the system that do not affect the kinetics of the system to describe the creation of stochastic fractals. In order to determine the fractal dimension of the support, we find it convenient to use the box-counting method. We may choose to associate each hidden conserved quantity with a set of points in $\Re^2$ space. In order to measure the set we can subdivide the space into small squares of sides δ_{m*}, where δ_{m*} is defined as

$$\delta_{m*} = \sqrt{\frac{M_{m*,4/m*}(t)}{M_{1,1}(t)}}, \tag{30}$$

such that $\mu_i(\delta_{m*})$ denotes the measure within the ith box. Obviously, when $s=4$ the measure will count the complete set of points in the plane as $\delta_{m*}\to 0$ and independent of $m*$. However, for $1<s\leq 4$, the measure will depend on the s value as well as on the $m*$ value, as we shall see now. For $1<s\leq 4$ and in the limit $\delta_{m*}\to 0$ the total number of boxes required to cover the set of points can be expressed as

$$\langle N(\delta_{m*})\rangle \sim \delta_{m*}^{-\gamma(m*,\beta_1,\beta_2)\{\sqrt{(\beta_2-\beta_1)^2+G}-(\beta_1+\beta_2+2)\}}$$
$$= \delta_{m*}^{-D_f(m*,\beta_1,\beta_2)}. \tag{31}$$

We define $A_\pm = m*(\beta_2+1) \pm 4/m*(\beta_1+1)$, to express $\gamma(m*,\beta_1,\beta_2)$ as

$$\gamma(m*,\beta_1,\beta_2) = \frac{2}{A_+ - \sqrt{A_-^2+G} - (\beta_1+\beta_2+2) + \sqrt{(\beta_2-\beta_2)^2+G}}. \tag{32}$$

The exponent D_f is the Hausdorff-Besicovitch dimension that measures the properties of the set of points. Note that $\gamma(m*,\beta_1,\beta_2) = \gamma(s/m*,\beta_1,\beta_2)$, and in the limit $m*\to\infty$,

$$\gamma(m*,\beta_1,\beta_2) = \frac{2}{\sqrt{(\beta_2-\beta_1)^2+G} - (\beta_1+\beta_2+2)}. \tag{33}$$

Moreover, when $s=4$, D_f is independent of $m*$ and individual β values, i.e., independent of the intensity of fragmentation. However, for $1<s<4$ and for fixed β_1,β_2 values, D_f increases monotonically against $(m*+4/m*)$ starting at a value when $(m*+4/m*)$ is minimum and saturates at 2 as $m*\to\infty$. Furthermore, when $\beta_1=\beta_2=\beta$ [21] D_f is independent of β.

B. Model *B*

As before we may choose to associate each hidden conserved quantity with a set of points in $\mathcal{R}^2$ space. This space can be subdivided into boxes of sides

$$\delta_{m*} = \sqrt{\frac{M_{m*,4-m*}(t)}{M_{1,1}(t)}}. \tag{34}$$

In the limit $\sigma_{m*}\to 0$ the total number of boxes required to cover the set of points can be expressed as

$$\langle N(t)\rangle \sim \delta_{m*}^{-\ln s/\ln 2}. \tag{35}$$

Unlike model A, here the fractal dimension is independent of the $m*$ value and of homogeneity indices. This reveals that a single scaling exponent D_f can describe such a self-similar structure.

V. MULTIFRACTALITY

A. Model *A*

It is important to realize that a single exponent D_f is not sufficient to characterize the present system under investigation. To show this we now express the quantity $M_{m,1}(t)$ in terms of the box length δ_{m*} as $\delta_{m*}\to 0$ to give

$$M_{m,1}(\delta_{m*}) \sim \delta_{m*}^{-\gamma(m*,\beta_1,\beta_2)\{\sqrt{[m(\beta_2+1)-(\beta_1+1)]^2+G}-m(\beta_2+1)-(\beta_1+1)\}}. \tag{36}$$

Obviously, the exponent of the above equation when $m=1$ gives the dimension of the measure of the support D_f.

From the definition of the moment we can write

$$M_{q,1}(t) = \int x^q n(x,t) d\ \ln x = \int e^{F(x,q,t)} d\ \ln x, \tag{37}$$

where $n(x,t) = \int dy\ f(x,y;t)$, $n(x,t)d\ln x$ is the number of branches characterized by x in the interval $[\ln x, \ln x + d\ln x]$ and $F(x,q,t) = \ln[n(x,t)] + q\ln x$. In the multifractal formalism the quantity $M_{q,1}(t)$ is often identified as the partition function motivated by the analogy with thermodynamics. Now, following the approach of Refs. [22,23], the integral in Eq. (37) can be evaluated by the steepest-descent method. If, say, $x*$ is the value for which $F(x,q,t)$ has a maximum value, then we have

$$\left.\frac{\partial \ln n(x,t)}{\partial \ln x}\right|_{x=x*} = -q. \tag{38}$$

In general, for each value of q there is a corresponding value of $x = x*(q)$, and so immediately one can write the following scaling ansatz:

$$x^* \sim A(q)\,\delta_{m*}^{-\alpha(q)}, \tag{39}$$

$$n(x^*,\delta_{m*}) \sim B(q)\,\delta_{m*}^{f(q)}, \tag{40}$$

since

$$M_{q,1}(\delta_{m*}) \sim \delta_{m*}^{-q\alpha(q)-f(q)}. \tag{41}$$

As q varies from $-\infty$ to ∞, x^* takes all the values depending on q values and hence-forth we call it x instead of x^*. We find it convenient to define the quantity

$$\nu = \frac{\ln x}{\ln x_{\max}} = \frac{\alpha(q)}{\alpha(\infty)}. \tag{42}$$

Therefore, for each value of ν there exists a corresponding value $q(x)$. From (40) we get

$$n(x,\delta_{m*}) \sim C(\nu)\,\delta_{m*}^{\Phi(\nu)}, \tag{43}$$

where $\Phi(\nu)=f(q(\nu))$ is the spectrum of the fractal subset and $C(\nu)=B(q(\nu))$. Using (39) and (40) we can also write

$$n(x) \sim C(\nu) x^{-\Phi(\nu)/\nu\alpha(\infty)}. \tag{44}$$

This scaling form expresses the fact that the fragmentation process can be partitioned into subsets when each is characterized by the value $\nu=\ln x/\ln x_{\max}$. Each subset has an independent fractal dimension given by $\Phi(\nu)$ and by the singularity exponent $\alpha(\infty)$. Scaling of this kind has also been found for the random resistor network and in the context of diffusion-limited aggregation.

We shall now attempt to find the explicit expressions for these exponents. In order to do this we write the d measure of the weighted box number as

$$M_{m,1}(d,\delta_{m*}) = \sum_i \mu_i^m \delta_{m*}^d = M_{m,1}(\delta_{m*})\,\delta_{m*}^d. \tag{45}$$

It follows from Eq. (43) that as $\delta_m^* \to 0$,

$$M_{m,1}(d,\delta_{m*}) \to \begin{cases} 0 & \text{if } d>\tau(m), \\ \infty & \text{if } d<\tau(m). \end{cases} \tag{46}$$

The critical value for which the measure tends to a finite value is called the mass exponent $d_c=\tau(m)$. Note that $d_c=\tau(1)$ is the dimension of the measure support. We thus see that $M_{m,1}(\delta_{m*})$ can be partioned into boxes of sides δ_{m*} such that the probabilities of $\mu_i(\delta_{m*})$ are normalized if we replace m by

$$1+(s-1)k \equiv q(k). \tag{47}$$

Hence we can write the weighted box number (the so-called partition function) as

$$M_{k,1}(\delta_{m*}) = \sum_{i=1} \mu_i^{k(m,\beta_1,\beta_2)}(\delta_{m*}) \sim \delta_{m*}^{-\tau(k,\beta_1,\beta_2)}. \tag{48}$$

Using (47) in (36) and then comparing with (48), we immediately can obtain the expression for $\tau(k)$, the mass exponent to be

$$\tau(k) = \gamma(m^*,\beta_1,\beta_2)\{\sqrt{[(\beta_2-\beta_1)+(s-1)(\beta_2+1)k]^2+G} - (\beta_2+1)(s-1)k-(\beta_1+\beta_2+2)\}, \tag{49}$$

which meets the essential requirement, namely, $\tau(0)$ is the dimension of the support and $\tau(1)=0$. We thus see that there exists a spectrum of mass exponent $\tau(k)$, which characterizes the distribution of the particle size. Nevertheless, the mass exponent $\tau(k)$ is nonlinear, which simply reveals that there exists a spectrum fractal subset for each support whether the support itself is fractal or not. To find the fractal subset we use the usual Legendre transform of the independent variables τ and k to the independent variable α and $f(\alpha)$:

$$\alpha(k) = -\frac{d\tau(k)}{dk}, \tag{50}$$

and

$$f(\alpha(k)) = k\alpha(k)+\tau(k). \tag{51}$$

These relations yield

$$\alpha(k) = \gamma(m^*,\beta_1,\beta_2)(s-1)(\beta_2+1)\left\{1 - \frac{(\beta_2-\beta_1)(\beta_2+1)(s-1)+(\beta_2+1)(s-1)k}{\sqrt{[(\beta_2-\beta_1)+(s-1)(\beta_2+1)k]^2+G}}\right\} \tag{52}$$

and

$$f(\alpha(k)) = \gamma(m^*,\beta_1,\beta_2)\left\{\sqrt{[(\beta_2-\beta_1)+(s-1)(\beta_2+1)k]^2+G}-(\beta_1+\beta_2+2) - \frac{(s-1)(\beta_2-\beta_1)(\beta_2+1)k+(s-1)^2(\beta_2+1)^2k^2}{\sqrt{[(\beta_2-\beta_1)+(s-1)(\beta_2+1)k]^2+G}}\right\}. \tag{53}$$

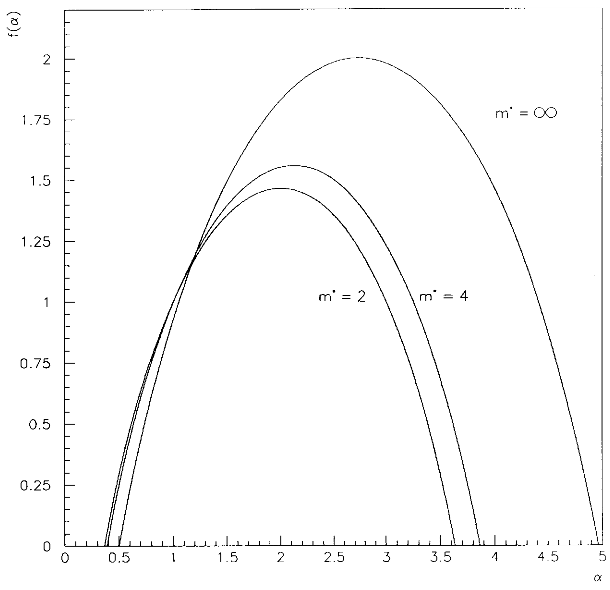

FIG. 1. Three of the f-α spectra for model A when $s=3$ and $\beta_1=\beta_2$. The three curves are for $m^*=2$, 4, and ∞ when $\beta_1=\beta_2$.

Note that all the exponents depend on the homogeneity indices if only $\beta_1 \neq \beta_2$. We find that if we choose homogeneity indices to be at any point in the shattering regime, the whole formalism and analysis breaks down. That is, in these regimes the moment does not show power-law behavior. Of course, one point on the phase boundary where all the phase boundary meets $(\beta_1=\beta_2=-1)$, the moments exhibit exponential behavior instead of power-law behavior. Physically the $f(\alpha(k))$ versus α curve simply suggests the existence of intertwined fractal subsets describing the measure support. The expression for $f(\alpha(k))$ is strictly convex in nature, as can be seen from Fig. 1. We find that when the system describes stochastic fractals there exists a hierarchy of fractal support D_f that depends on m^* for a fixed value of homogeneity indices. In Fig. 1 we plot $f(\alpha(k))$ against $\alpha(k)$ for three different m^* values when $\beta_1=\beta_2$ to show that there can be infinitely many possible supports on which a measure can be distributed when fragments are removed from the system at each event. It is to see that the preceding analysis can be repeated for $M_{1,q}(t)$ as a partition function since the definition of moments are symmetric in their variables.

B. Model *B*

Doing a similar calculation for the second model one can immediately find that despite the fact that there exists infinitely many hidden conserved quantities, the fractal dimension is independent of m^*'s and homogeneity indices, giving a unique measurement, not a spectrum, as it should be,

$$D_f=\frac{\ln s}{\ln 2}, \tag{54}$$

where $D_f=f(\alpha)$ and $\tau(k)=D_f(1-k)$ to reveal that for $1<s<4$, the system describes the self-similar fractals and hence confirms the existence of scaling. In this case, the same scaling exponent D_f describes the asymptotic behavior of all the characteristic lengths $[L^m(t)=\sqrt[m]{M_{m,1}(t)/M_{1,2}(t)}]$ i.e., independent of the definition of the characteristic length. Evidently, there is a constant gap between the consecutive moments. Note that for $s=4$ we again recover the full set of points in the plane $(D_f=2)$.

VI. CONCLUSIONS

Model B we consider as a supporting model since it is simple and its important aspect is known from Ref. [4]. It is model A that is of primary concern in understanding what shattering means when there is more than one dynamical variable in the system. We find that instead of a unique phase boundary for the shattering transition we find a multiple phase boundary. We attempt to explain this surprising feature by the idea of multifractal formalism. In the case of model A the support can be partitioned into infinitely many subsets of their own fractal dimensions. That is, each subset scales with different kinetic exponents *vis-a-vis* different fractal dimensions. Consequently, each subset has its own phase boundary for which the corresponding subset of the support goes into the shattering transition.

When the system describes the fragmentation process $(s=4)$, we find that the system gives a unique measure support $(D_f=2)$ on which subsets can be distributed. However, any observable fluctuates strongly from one realization to the other. Although each realization is statistically self-similar in these fluctuations, it means that averaged quantities of any observable can be measured with a reasonable accuracy only through ensemble averaging. That is, a single experiment for a longer time will not give any averaged quantities with good accuracy, but a large number of independent experiments are required, which is a very important property to know for real or numerical experiments. But when describing stochastic fractals, one associates pictures of wildly varying probabilities of the measure, since at each realization the dimension of the support can be different. This reflects the fact that in the case when a system describes stochastic fractals, the entropy of the system has one more source than in the fragmentation process. This extra source arises due to the competition among the fractal support for different m^* in a given experiment. Note that when three fragments are removed from the system at each time event $(s=1)$, the dimension of the measure support (D_f) is zero where the measure can be distributed.

It is interesting to notice the connection between model A for $1<s<4$ and the random sequential adsorption (RSA). For $s=2$ this model has been discussed in the context of the RSA of needles in Ref. [30]. We remark that if objects are of a definite size and once deposited are clamped in their position, the resultant configurations in the long-time limit are highly nonergodic with a strong non-Markovian nature. Such a system has a universal feature described by the jamming limit, which is less than random close packing. Our model for $1<s<4$ can be thought of as the deposition of $4-s$ particles at each time step. The difference between the true RSA and our system is that for $1<s<4$ particles are deposited in a rather restricted set of positions and the size of the particles to be deposited is determined by the available space. Consequently, the system gains the ergodic nature with which the scaling is possible, and in the long time instead of reaching a jamming limit it shows power-law behavior. Although the present model cannot be described by the jamming limit, the resultant structure in the long time can be described geometrically as stochastic fractals with self-similar features characterized by fractal dimension D_f. The model we discuss could be a potential candidate to describe some features of RSA since in the long time the resultant distribution is indistinguishable from the random deposition of a mixture of particles of rectangles.

The origin of the occurrence of multifractality in different physical systems is yet not fully understood, despite its importance in many physical systems. The two models we discuss give us the opportunity of finding the reason why one needs an infinite number of independent exponents to characterize the scaling relation in model A while the later model describes simple scaling. These two models can be very good candidates to look for the answer since both models give infinitely many conserved quantities, and both have been derived in two dimensions yet show different behaviors. To find the answer we need to go back to the nature of the model itself and search for the things we lost in moving from model A to the second model. In model A we had stochastic homogeneity, which implies that the fragmentation of an object possesses an ergodic probability distribution. In this model, two infinitely long and orthogonal cracks are placed on the objects independently and parallel to the sides, i.e., they can pass through any point in Euclidean space. Thus at each fragmentation event, the four fragments can be of any shape, provided their total area is conserved. Thus during the fragmentation process the dynamical variable size is influenced by shape and consequently shape is a dynamical quantity. While model B describes two infinitely long and orthogonal cracks that are allowed to be placed-only at the middle of the objects to successfully produce four equal-sized fragments, it implies that the size is no longer influenced by shape, i.e., shape is determined by the initial condition. That has been shown in Ref. [4] by solving the rate equation explicitly. Note that it is one of the infinite possibilities of the former model. Thus if there is a mixture of particles of different size and shape, and if all the fragments are equally likely to be picked from the mixture, in the second model once a fragment with definite shape is picked for fragmentation, this will only produce fragments of that shape. Thus it is the broken ergodicity in space of shape that causes the absence of multiscaling in the second model. One often finds it convenient to make an analogy with the thermodynamics by identifying $M_{q,1}(\delta_{m*})$ as the partition function. We conclude with the remark that perhaps the origin of multifractal phenomena associated with the system and the underlying physics is governed by more than one intriguing dynamical variable.

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